

ON VOLUME GROWTH OF GRADIENT STEADY RICCI SOLITONS

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Abstract

In this paper we study volume growth of gradient steady Ricci solitons. We show that if the potential function satisfies a uniform condition, then the soliton has at most Euclidean volume growth.

1 Introduction

(M^n, g) is a gradient Ricci soliton if there is smooth function $f : M \rightarrow \mathbb{R}$ and constant $\lambda \in \mathbb{R}$ such that

$$\text{Ric} + \text{Hess}f = \lambda g. \quad (1.1)$$

f will be referred as the potential function. The soliton is called shrinking, steady, expanding when $\lambda > 0, \lambda = 0, \lambda < 0$ respectively.

Ricci solitons are self-similar solutions of the Ricci flow, and play an important role in the study of singularity formation. They are also natural extensions of Einstein manifolds, and special cases of smooth metric measure spaces.

Volume growth of gradient Ricci solitons is of particular interest to mathematicians. Estimate of the potential functions plays an important role in the study of volume growth. In [7], Hamilton proved the following identity for gradient Ricci solitons,

$$R + |\nabla f|^2 - 2\lambda f = \Lambda,$$

where Λ is a constant, and R is the scalar curvature.

For gradient shrinking Ricci solitons, the answer is complete. Perelman [10] and Cao-Zhou [4] proved that f always grows quadratically. Cao-Zhou [4] further proved that any gradient Ricci shrinking soliton has at most Euclidean volume growth. Recently, Munteanu-Wang [9] proved that any gradient Ricci shrinking soliton has at least linear volume growth.

For gradient steady Ricci solitons, B.-L. Chen [5] proved that $R \geq 0$. Hence $\Lambda \geq 0$ and equal to zero if and only if f is constant and (M, g) is Ricci flat. When $\Lambda > 0$ we can assume $\Lambda = 1$ after scaling, i.e.

$$R + |\nabla f|^2 = 1. \quad (1.2)$$

Combine with the trace of steady Ricci soliton equation $R + \Delta f = 0$, we have

$$\Delta f - |\nabla f|^2 = -1. \quad (1.3)$$

Therefore f has no local minimum. (1.2) and that $R \geq 0$ also gives $|\nabla f| \leq 1$. Namely f decays at most linearly.

Cao-Chen [3] proved that f decays linearly when Ricci curvature is positive and R attains its maximal at some point. However the simple example of \mathbb{R}^2 with the canonical metric g_0 and $f(x) = x_1$ shows that this is not the case: f is constant along x_2 direction.

Note that Riemannian product of any two steady gradient Ricci solitons is still a steady gradient Ricci soliton. Hence a steady gradient Ricci soliton multiply with a trivial one (f is a constant) will have constant direction. Though one can make product of two shrinking ones, but all trivial shrinking ones are compact. So it will not give a constant direction by taking a product. Munteanu-Sesum [8] and the second author [13] independently showed that the infimum of f does decay linearly. In fact

$$-r \leq \inf_{y \in \partial B(x,r)} f(y) - f(x) \leq -r + \sqrt{2n}(\sqrt{r} + 1), \quad r \gg 1. \quad (1.4)$$

In particular, $\liminf_{y \rightarrow \infty} R(y) = 0$, see also [6, 1].

We note that among all known examples of steady gradient Ricci solitons, the infimum of f is like $-r + O(\ln r)$. See the survey article [2] for a list of examples. One naturally asks if one can improve the second order term in (1.4) from \sqrt{r} to $\ln r$. We shows this is indeed the case for a large class of steady gradient Ricci solitons. To study the second order term, write the potential function in polar coordinate:

$$f(r, \theta) = -r + \phi(r, \theta)$$

where $r(\cdot) = d(x, \cdot)$ for some $x \in M^n$, $\theta \in S^{n-1}$. Without loss of generality we assume $\phi(0, \theta) = 0$ by adding a constant to f . Since $f(r) \geq -r$ and $|\nabla f| \leq 1$, $\phi(r) \geq 0$ and $\phi(r, \theta)$ is nondecreasing in r for any fixed θ . We show that the estimate (1.4) can be improved to $\ln r$, if ϕ at one direction is comparable to minimum of ϕ among all spherical directions for all r large. Namely,

Theorem 1.1 *Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying (1.2). Assume that there exist $\theta_0 \in S^{n-1}$, and constants $C_1 \geq 0, C_2 \geq 0$ such that*

$$\int_0^r (\phi(r, \theta_0) - \phi(t, \theta_0)) dt \leq C_1 \min_{\theta \in S^{n-1}} \int_0^r (\phi(r, \theta) - \phi(t, \theta)) dt + C_2 r \quad (1.5)$$

for sufficiently large r . Then for any $x \in M^n$, there exist constants $C \geq 0, r_0 > 0$ such that for $r \geq r_0$,

$$-r \leq \inf_{y \in \partial B(x,r)} f(y) - f(x) \leq -r + \left(\frac{n}{2}C_1 + C_2\right) \ln r + C. \quad (1.6)$$

All known examples of gradient steady Ricci solitons satisfy the condition (1.5). We suspect that the estimate (1.6) holds for all gradient steady Ricci solitons.

In [8], Munteanu and Sesum proved that any gradient steady Ricci soliton has at least linear volume growth, and at most growth rate of $e^{\sqrt{r}}$. We show that if the potential function satisfies a uniform condition in the spherical directions, then the gradient steady Ricci soliton has at most Euclidean volume growth.

Theorem 1.2 *Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying (1.2). Assume that there exist constants $C_1, C_2 \geq 0$ such that*

$$\max_{\theta \in S^{n-1}} \int_0^r \phi(r, \theta) - \phi(t, \theta) dt \leq C_1 \min_{\theta \in S^{n-1}} \int_0^r \phi(r, \theta) - \phi(t, \theta) dt + C_2 r \quad (1.7)$$

for sufficiently large r . Then for any $x \in M^n$, there exists $r_0 > 0$, for any $r \geq r_0$,

$$-r \leq f(y) - f(x) \leq -r + C \ln r. \quad (1.8)$$

for any $y \in \partial B(x, r)$. Moreover, the soliton has at most Euclidean volume growth, i.e. for any $x \in M^n$, there exists $r_0 > 0$, for any $r \geq r_0$,

$$\text{Vol}(B(x, r)) \leq Cr^n.$$

If in addition $\phi(r) \geq \delta \ln r$ for large r , then

$$\text{Vol}(B(x, r)) \leq Cr^{n-\delta}.$$

Remark 1.3 (1). If ϕ increases uniformly along all spherical directions, i.e. $\max_{\theta} \frac{\partial \phi}{\partial r} \leq C \min_{\theta} \frac{\partial \phi}{\partial r}$, where $\theta \in S^{n-1}$, then ϕ satisfies (1.7) with $C_1 = C$, $C_2 = 0$.

(2). Theorem 1.2 can be considered as an analogue of volume growth theorem for gradient shrinking Ricci solitons of [4]. For gradient shrinking Ricci solitons, the potential function automatically satisfies a uniform condition [4]; for gradient steady Ricci solitons, we need to impose a uniform condition.

(3). If the soliton is rectifiable (see [11]), i.e. f is the distance function from a set, then ϕ satisfies (1.7) with $C_1 = 1$ if the set is bounded (this is the case with all nonproduct examples).

To prove the results, the following estimate for ϕ which holds for all gradient steady Ricci solitons is the key.

Proposition 1.4 Let (M^n, g, f) be a complete gradient steady Ricci soliton satisfying (1.2). Then

$$\min_{y \in \partial B(x, r)} \int_0^r (\phi(y) - \phi(t)) dt \leq \frac{n}{2}(r + \sqrt{r}) + o\left(\frac{1}{r}\right). \quad (1.9)$$

This estimate improves the estimate in [13]. In the next section we derive a volume comparison for the solitons by adapting the volume comparison for smooth metric measure in [12]. Then we prove Proposition 1.4 by combining with equation (1.3). In Section 3 we prove the main theorems using this estimate and an ODE.

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2 The Preliminary Estimate

In this section we prove Proposition 1.4 by applying an weighted volume comparison argument for smooth metric measure spaces as in [12, 13].

Recall a smooth metric measure space is a triple $(M^n, g, e^{-f} d\text{vol}_g)$, where (M^n, g) is a smooth Riemannian manifold, and $f : M^n \rightarrow \mathbb{R}$ is a smooth function. Write the volume element in polar coordinate $d\text{vol} = J(r, \theta) dr d\theta$. Define the weighted volume element $J_f(r, \theta) = e^{-f} J(r, \theta)$, and weighted volume $\text{vol}_f B(x, r) = \int_{B(x, r)} e^{-f} d\text{vol}$.

Wei and Wylie [12] obtained the following f -volume comparison theorem for smooth metric measure spaces,

Theorem 2.1 (*f-volume comparison*)

Suppose $(M^n, g, e^{-f} d\text{vol})$ is a smooth metric measure space with $\text{Ric}_f \geq (n-1)H$. Fix $x \in M$. If $|f| \leq \Lambda$, then for $R \geq r > 0$ ($R \leq \pi/4\sqrt{H}$ if $H > 0$),

$$\frac{V_f(B_R(x))}{V_f(B_r(x))} \leq \frac{V_H^{n+4\Lambda}(B_R)}{V_H^{n+4\Lambda}(B_r)}.$$

Where $V_H^n(B_r)$ is the volume of the ball of radius r in M_H^n , the simply connected model space of dimension n with constant sectional curvature H .

One observes that the dimension of the model space in the volume comparison depends on the potential function f . A further investigation of the dimension will lead to Proposition 1.4.

Denote $m_f = (\ln J_f)'$, the f -mean curvature. For $0 < r_1 \leq r_2$, let $A(x, r_1, r_2) = \{y | r_1 \leq d(x, y) \leq r_2\}$ be the annulus, and

$$a = \min_{y \in A(x, r_1, r_2)} \frac{2}{r(y)^2} \int_0^{r(y)} (\phi(y) - \phi(t)) dt.$$

Clearly $a \geq 0$. By (1.4), we have $a \leq \frac{C}{\sqrt{r_1}}$ for $r_1 \gg 1$. For the rest we assume $r_1 \gg 1$ and therefore we can assume $a < 1$.

Proposition 2.2 *For gradient steady Ricci soliton, we have*

$$m_f(r, \theta) \leq \frac{n-1}{r} + 1 - \frac{2}{r^2} \int_0^r [\phi(r, \theta) - \phi(t, \theta)] dt \leq \frac{n-1}{r} + 1. \quad (2.10)$$

and

$$\frac{\text{Vol}_f(\partial B(x, r_2))}{\text{Vol}_f(A(x, r_1, r_2))} \leq \frac{\frac{n}{r_2} + 1 - a}{1 - (\frac{r_1}{r_2})^{n+(1-a)r_2}}. \quad (2.11)$$

Proof: For smooth metric space (M^n, g, f) with $\text{Ric}_f \geq 0$, recall the following estimate for m_f from [12, (3.19)],

$$m_f(r, \theta) \leq \frac{n-1}{r} + \frac{2}{r^2} \int_0^r (f(t) - f(r)) dt.$$

Plug in $f = -r + \phi$ gives (2.10).

Now let

$$\overline{m}(r) = \begin{cases} \frac{n-1}{r} + 1 & r \leq r_1 \\ \frac{n-1+(1-a)r_2}{r} & r_1 < r \leq r_2 \end{cases},$$

then

$$m_f(r) \leq \overline{m}(r) \quad \text{for } 0 < r \leq r_2. \quad (2.12)$$

Let $\overline{A}(r) = e^{\int_0^r \overline{m}(t) dt}$ and $\overline{V}(r_0, r) = \int_{r_0}^r \overline{A}(t) dt$. From the mean curvature relation (2.12), we have $\left(\frac{A_f}{A}\right)' \leq 0$, therefore

$$\frac{\text{Vol}_f(\partial B(x, r_2))}{\text{Vol}_f(A(x, r_1, r_2))} \leq \frac{\overline{A}(r_2)}{\overline{V}(r_1, r_2)},$$

We compute

$$\begin{aligned} \frac{\overline{A}(r_2)}{\overline{V}(r_1, r_2)} &= \frac{e^{\int_0^{r_2} \overline{m}(t) dt}}{\int_{r_1}^{r_2} e^{\int_0^s \overline{m}(t) dt} ds} = \frac{e^{\int_{r_1}^{r_2} \overline{m}(t) dt}}{\int_{r_1}^{r_2} e^{\int_{r_1}^s \overline{m}(t) dt} ds} \\ &= \frac{(r_2/r_1)^{n-1+(1-a)r_2}}{\int_{r_1}^{r_2} (s/r_1)^{n-1+(1-a)r_2} ds} \\ &= \frac{\frac{n}{r_2} + 1 - a}{1 - (\frac{r_1}{r_2})^{n+(1-a)r_2}}. \end{aligned}$$

This gives (2.11). ■

Proof of Proposition 1.4.

Integrating (1.3) and using $|\nabla f| \leq 1$, we have, for any $x \in M$,

$$\begin{aligned} \int_{B(x,r)} 1 \cdot e^{-f} d\text{vol} &= - \int_{B(x,r)} (\Delta f - |\nabla f|^2) \cdot e^{-f} d\text{vol} \\ &= - \int_{\partial B(x,r)} \frac{\partial f}{\partial n} e^{-f} d\text{vol} \leq \int_{\partial B(x,r)} e^{-f} d\text{vol}. \end{aligned}$$

Therefore

$$\frac{\text{Vol}_f(\partial B(x, r))}{\text{Vol}_f(B(x, r))} \geq 1. \quad (2.13)$$

Combining (2.11) and (2.13) we have

$$a \leq \frac{n}{r_2} + \left(\frac{r_1}{r_2}\right)^{n+(1-a)r_2}.$$

Let $r_1 = r, r_2 = r + \sqrt{r}$, then $r_1/r_2 = (1 + \frac{1}{\sqrt{r}})^{-1}$. When r is large,

$$\left(1 + \frac{1}{\sqrt{r}}\right)^{-(n+(1-a)(r+\sqrt{r}))} = O(e^{-(1-a)\sqrt{r}}).$$

Therefore, for all r large enough,

$$a = \min_{y \in A(x, r, r+\sqrt{r})} \frac{2}{r(y)^2} \int_0^{r(y)} (\phi(y) - \phi(t)) dt \leq \frac{n}{r + \sqrt{r}} + O(e^{-(1-a)\sqrt{r}}).$$

Suppose the minimum in above is attained at $y_0 = (r_0, \theta_1)$ with $r \leq r_0 \leq r + \sqrt{r}$. Then

$$\begin{aligned} \min_{y \in \partial B(x, r)} \int_0^r (\phi(y) - \phi(t)) dt &\leq \int_0^{r_0} (\phi(y_0) - \phi(t)) dt \\ &\leq \frac{r_0^2}{2} \left(\frac{n}{r + \sqrt{r}} + O(e^{-(1-a)\sqrt{r}}) \right) \\ &\leq \frac{n}{2} (r + \sqrt{r}) + o\left(\frac{1}{r}\right). \end{aligned}$$

■

3 Proof of Main Results

Proof of Theorem 1.1. From (1.9) and (1.5), we have

$$\int_0^r [\phi(r, \theta_0) - \phi(t, \theta_0)] dt \leq \frac{nC_1}{2} (r + \sqrt{r}) + C_2 r + o\left(\frac{1}{r}\right). \quad (3.14)$$

For simplicity when there is no confusion we omit θ_0 in the function. Let $\Phi(r) = \int_0^r \phi(t) dt$, then the above inequality can be written as,

$$\Phi'(r) - \frac{1}{r} \Phi(r) \leq \frac{nC_1}{2} + C_2 + O\left(\frac{1}{\sqrt{r}}\right). \quad (3.15)$$

Multiply by the integrating factor $\frac{1}{r}$ and integrate from some fixed $t_0 \gg 1$ to r , we get

$$\frac{\Phi(r)}{r} \leq \left(\frac{nC_1}{2} + C_2 \right) \ln r + C_3.$$

So we have

$$\begin{aligned} \phi(r, \theta_0) &= \Phi'(r, \theta_0) \leq \frac{\Phi(r, \theta_0)}{r} + \frac{nC_1}{2} + C_2 + O\left(\frac{1}{\sqrt{r}}\right) \\ &\leq \left(\frac{nC_1}{2} + C_2 \right) \ln r + C_4 \\ f(r, \theta_0) &= -r + \phi(r, \theta_0) \leq -r + \left(\frac{nC_1}{2} + C_2 \right) \ln r + C_4. \end{aligned}$$

This gives (1.6). ■

Proof of Theorem 1.2 From (1.9) and (1.7), we have

$$\int_0^r [\phi(r, \theta) - \phi(t, \theta)] dt \leq \frac{nC_1}{2}(r + \sqrt{r}) + C_2 r + o\left(\frac{1}{r}\right),$$

for all $\theta \in S^{n-1}$. Therefore (1.6) holds for all y . Namely, for all $y \in \partial B_r(x)$,

$$-r \leq f(y) - f(x) \leq -r + \left(\frac{nC_1}{2} + C_2 \right) \ln r + C_4.$$

By (2.10), for all $r > 0$,

$$\begin{aligned} \frac{\partial}{\partial r} \ln J &= m_f(r) + \langle \nabla f, \nabla r \rangle \\ &\leq \frac{n-1}{r} + 1 - \frac{2}{r} \phi(r) + \frac{2}{r^2} \int_0^r \phi(t) dt + \langle \nabla f, \nabla r \rangle. \end{aligned}$$

Integrate from 1 to r and do integration by part for the double integral, we get

$$\begin{aligned} \ln J(r) - \ln J(1) &\leq (n-1) \ln r + (r-1) - \int_1^r \frac{2}{s} \phi(s) ds \\ &\quad + \left(-\frac{2}{s} \int_0^s \phi(t) dt \right) \Big|_1^r + \int_1^r \frac{2}{s} \phi(s) ds + f(r) - f(1) \\ &= (n-1) \ln r + \phi(r) - \frac{2}{r} \int_0^r \phi(t) dt + 2 \int_0^1 \phi(t) dt - f(1) \\ &= (n-1) \ln r - \phi(r) + 2 \left(\phi(r) - \frac{1}{r} \int_0^r \phi(t) dt \right) + 2 \int_0^1 \phi(t) dt - f(1). \end{aligned} \tag{3.16}$$

Using (3.15) we have, for r large,

$$\ln J(r) \leq (n-1) \ln r - \phi(r) + C \leq (n-1) \ln r + C.$$

Hence

$$J(r) \leq e^C e^{(n-1) \ln r} = e^C r^{n-1},$$

and the volume of geodesic ball centering at x satisfies

$$\text{Vol}(B(x, r)) \leq C' r^n.$$

If further $\phi(s) \geq \delta \ln s$, then we have

$$J(r) \leq C r^{n-1} \exp(-\phi(r)) \leq C r^{n-\delta-1},$$

therefore the volume growth is strictly less than Euclidean volume growth,

$$\text{Vol}(B(x, r)) \leq C r^{n-\delta}. \quad (3.17)$$

■

For general gradient steady Ricci solitons, the estimate of potential function can be reduced to the following

Question 3.1 *Suppose ϕ is nondecreasing along any minimal geodesic starting from x . Assume that for r sufficiently large, $\inf_{y \in \partial B(x, r)} \phi(y) \leq C\sqrt{r}$ and*

$$\inf_{y \in \partial B(x, r)} \int_1^r [\phi(y) - \phi(\gamma_y(t))] dt \leq \frac{nr}{2}.$$

Does the following hold?

$$\inf_{y \in \partial B(x, r)} \phi(y) \leq C \ln r?$$

Remark 3.2 *From (3.16), we see that if*

$$-r \leq f(y) - f(x) \leq -r + C \ln r.$$

for $y \in \partial B(x, r)$, then for any $x \in M^n$, there exists $r_0 > 0$ such that for any $r \geq r_0$,

$$\text{Vol}(B(x, r)) \leq C' r^{n+C}.$$

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